

DESCRIPTIVE PROXIMITIES I: PROPERTIES AND INTERPLAY BETWEEN CLASSICAL PROXIMITIES AND OVERLAP

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Dedicated to the Memory of Som Naimpally

ABSTRACT. The theory of descriptive nearness is usually adopted when dealing with sets that share some common properties even when the sets are not spatially close, *i.e.*, the sets have no members in common. Set description results from the use of probe functions to define feature vectors that describe a set and the nearness of sets is given by their proximities. A probe on a non-empty set X is a real-valued function $\Phi : X \rightarrow \mathbb{R}^n$, where $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$. We establish a connection between relations on an object space X and relations on the feature space $\Phi(X)$. Having as starting point the Peters proximity, two sets are *descriptively near*, if and only if their descriptions intersect. In this paper, we construct a theoretical approach to a more visual form of proximity, namely, descriptive proximity, which has a broad spectrum of applications. We organize descriptive proximities on two different levels: weaker or stronger than the Peters proximity. We analyze the properties and interplay between descriptions on one side and classical proximities and overlap relations on the other side.

1. INTRODUCTION

This article carries forward recent work on proximities [17, 18, 21, 22, 6].

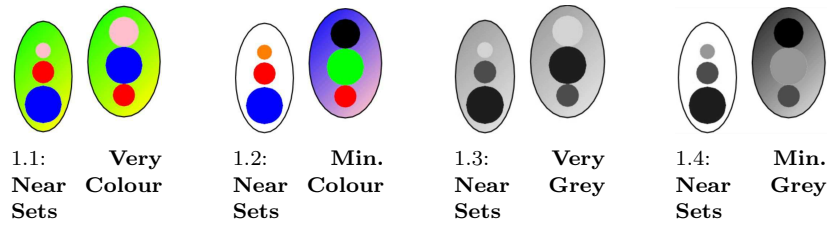


FIGURE 1. Descriptively near sets via colour or greyscale intensity

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Pivotal in this paper is the notion of a probe used to represent descriptions and proximities. A *probe* on a non-empty set X is real-valued function $\Phi : X \rightarrow \mathbb{R}^n$, where $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ and each ϕ_i represents the measurement of a particular feature of an object $x \in X$ [15] (see also [14]). We establish a connection between relations on the object space X and relations on the feature space $\Phi(X)$. Usually, probe functions describe or codify physical features and act like "sensors" in extracting characteristic feature values from the objects. The theory of descriptive nearness [16] is usually adopted when dealing with subsets that share some common properties even though the subsets are not spatially close.

Each pair of ovals in Fig. 1.1 and Fig. 1.2 contain circular-shaped coloured segments. Each segment in the ovals corresponds to an equivalence class, where all pixels in the class have matching descriptions, *i.e.*, pixels with matching colours. For the ovals in Fig. 1.1 and Fig. 1.2, we observe that the sets are not spatially *near*, but they can be considered near viewed in terms of colour intensities. Again, for example, the ovals in Fig. 1.3 and Fig. 1.4 contain segments that correspond to equivalence classes containing pixels with matching greyscale intensities. The ovals in Fig. 1.3 and Fig. 1.4 are descriptively near sets, since the equivalence classes contain matching greylevels. Moreover, we can also tell if they are more or less *near*. In the sequel, we will express these ideas of resemblance in mathematical terms.

We talk about *non-abstract points* when points have locations and features that can be measured. The description-based theory is particularly relevant when we want to focus on some distinguishing characteristics of sets of non-abstract points. For example, if we take a picture element x in a digital image, we can consider graylevel intensity or colour of x . In general, we define as a *probe* an n real valued function $\Phi : X \rightarrow \mathbb{R}^n$, where $\Phi(x) = (\phi_1(x), \dots, \phi_n(x))$ and each ϕ_i represents the measurement of a particular feature. So, $\Phi(x)$ is a feature vector containing numbers representing feature values extracted from x . And $\Phi(x)$ is also called *description* or *codification* of x . Of course, nearness or apartness depends essentially on the selected features that are compared.

J.F. Peters [16, §1.19] made the first fusion of description with proximity by introducing the notion of descriptive intersection of two sets:

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \Phi(A), \Phi(x) \in \Phi(B)\}$$

and by declaring two sets *descriptively near*, if and only if their descriptive intersection is non empty or equivalently, if and only if their descriptions intersect. That is the first step in passing from the classical spatial proximity to the more visual descriptive proximity. The new point of view is a really different approach to proximity which has a broad spectrum of applications. The Peters proximity, which we will denote as π_Φ , is the Φ -pullback of the set-intersection. By replacing the set-intersection with the descriptive intersection, we construct a theoretical approach to the more visual form of proximity, namely, descriptive proximity (denoted by δ_Φ). We organize descriptive proximities in two different levels: weaker ($A\delta_\Phi B \Rightarrow A \underset{\Phi}{\cap} B \neq \emptyset$) or stronger ($A \underset{\Phi}{\cap} B \neq \emptyset \Rightarrow A\delta_\Phi B$) than the Peters proximity. In both cases, we find a natural underlying topology. That is, descriptive intersection can be analyzed from the following two different perspectives: as the finest classical proximity, the discrete proximity, but also as the weakest overlapping relation, we exhibit significant examples of descriptive proximities weaker than the

Peters proximity by following two different options: the proximal approach and the overlapping approach.

1.1. Background of classical proximities.

We draw our reference from Naimpally-Di Concilio [4, 13] and are essentially interested in the simplest example of proximities, namely, Lodato proximities [9, 10, 11] which guarantee the existence of a natural underlying topology.

Definition 1.1 (Lodato). *Let X be a nonempty set. A Lodato proximity δ is a relation on $\mathcal{P}(X)$, the collection of all subsets of X , which satisfies the following properties for all subsets A, B, C of X :*

$$P_0): A \delta B \Rightarrow A \neq \emptyset \text{ and } B \neq \emptyset$$

$$P_1): A \delta B \Leftrightarrow B \delta A$$

$$P_2): A \cap B \neq \emptyset \Rightarrow A \delta B$$

$$P_3): A \delta (B \cup C) \Leftrightarrow A \delta B \text{ or } A \delta C$$

$$P_4): A \delta B \text{ and } \{b\} \delta C \text{ for each } b \in B \Rightarrow A \delta C$$

Further δ is separated, if

$$P_5) \{x\} \delta \{y\} \Rightarrow x = y.$$

When we write $A \delta B$, we read "A is near to B", while when we write $A \not\delta B$ we read "A is far from B". A relation δ which satisfies only $P_0) - P_3)$ is called a Čech [25] or basic proximity.

With any basic proximity one can associate a closure operator, cl_δ , by defining as closure of any subset A of X :

$$\text{cl}_\delta A = \{x \in X : \{x\} \delta A\}.$$

Definition 1.2. *An EF-proximity [2, 3] is a relation on $\mathcal{P}(X)$ which satisfies $P_0)$ through $P_3)$ and in addition the property:*

$$(EF) \quad A \not\delta B \Rightarrow \exists E \subset X \text{ such that } A \not\delta E \text{ and } X \setminus E \not\delta B$$

which can be formulated equivalently as:

$$(EF1) \quad A \not\delta B \Rightarrow \exists C, D \subset X, C \cup D = X \text{ such that } A \not\delta C \text{ and } D \not\delta B.$$

Since the EF-property is stronger than the Lodato property, every EF-proximity is indeed a Lodato proximity.

The following remarkable properties reveals the potentialities of Lodato proximity. When δ is a Lodato proximity, then:

Property.1 The associated closure operator cl_δ is a Kuratowski operator [7, 8]. Hence, every Lodato proximity space (X, δ) determines an associated topology $\tau(\delta)$ whose closed sets are just the subsets which agree with their own closures.

Property.2 Furthermore, for each subsets A, B :

$$A \delta B \iff \text{cl}_\delta A \delta \text{cl}_\delta B.$$

If (X, τ) is a topological space, we say that it admits a compatible Lodato proximity if there is a Lodato proximity δ on X such that $\tau = \tau(\delta)$. A question arises when a topological space has a compatible Lodato proximity. This happens when the space satisfies the R_0 -separation property, i.e. $x \in \text{cl}\{y\} \iff y \in \text{cl}\{x\}$. In fact, every R_0 topological space (X, τ) admits as a compatible Lodato proximity δ_0 given by:

$$A \delta_0 B \Leftrightarrow \text{cl}A \cap \text{cl}B \neq \emptyset. \text{ (Fine Lodato proximity [13])}$$

On the other hand, a topological space has a compatible EF-proximity if and only if it is a completely regular topological space [4, 26]. Recall that a topological space is completely regular iff whenever A is a closed set and $x \notin A$, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$ [26].

- Any Lodato T_1 (EF + T_2) proximity becomes *spatial* by a T_1 (T_2) compactification procedure.

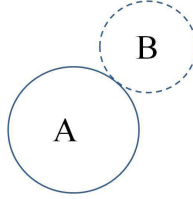


FIGURE 2. Overlapping Sets: $A \delta B$

1.2. Examples.

Example 1.3. Consider \mathbb{R}^2 endowed with the Euclidean topology and the sets in Fig. 2. A is an open disk while B is a closed disk. They are near in the fine Lodato proximity but they are far in the discrete proximity. ■

Example 1.4. Discrete Proximity on a Nonempty Set.

Let $A, B \subset X$. For a discrete proximity relation between A and B , we have $A \delta B \Leftrightarrow A \cap B \neq \emptyset$. This discrete proximity is a separated EF-proximity [§2.1][4]. ■

- From a spatial point of view, proximity appears as a generalization of the set-intersection. The discrete proximity from Example 1.4 gives rise to a discrete topology.
- A pivotal EF-proximity is the *metric proximity* δ_d associated with a metric space (X, d) defined by considering the gap between two sets in a metric space ($d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ or ∞ if A or B is empty) and by putting:

$$A \delta_d B \Leftrightarrow d(A, B) = 0.$$

That is, A and B are δ_d -near iff they *either intersect or are asymptotic*: for each natural number n there is a point a_n in A and a point b_n in B such that $d(a_n, b_n) < \frac{1}{n}$.

- *Fine Lodato proximity* δ_0 on a topological space is defined as follows:

$$A \delta_0 B \Leftrightarrow \text{cl}A \cap \text{cl}B \neq \emptyset.$$

The proximity δ_0 is the finest Lodato proximity compatible with a given topology.

- *Functionally indistinguishable proximity* δ_F on a completely regular space [4, §2.1, p.94].

$$A \delta_F B \Leftrightarrow \text{there is a continuous function } f : X \rightarrow [0, 1] : f(A) = 0, f(B) = 1.$$

The functionally indistinguishable proximity on a completely regular space X is an EF-proximity, which is further the finest EF-proximity compatible with X . Moreover, δ_F coincides with the fine Lodato proximity if and only if X is normal.

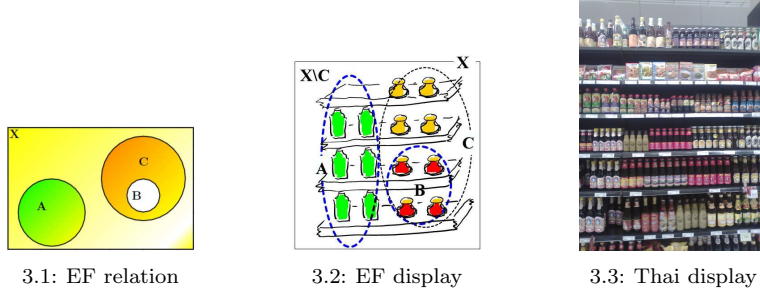


FIGURE 3. Sample EF-Relationships

Example 1.5. Descriptive EF Proximity Relation [12].

Let $A, C \subset X, B \subset C$ and let C^c be the complement of C . A descriptive EF proximity (denoted by ϕ_Φ) has the following property:

$$A \phi_\Phi B \Leftrightarrow A \phi_\Phi C \text{ and } B \phi_\Phi C^c.$$

A representation of this descriptive EF proximity relation is shown in Fig. 3.1. The import of an EF-proximity relation is extended rather handily to visual displays of products in a supermarket (see, e.g., Fig. 3.3). The sets of bottles that have an underlying EF-proximity to each other is shown conceptually in the sets in Fig. 3.2. The basic idea with this application of topology is to extend the normal practice in the vertical and horizontal arrangements of similar products with a consideration of the topological structure that results when remote sets are also taken into account, representing the relations between these remote sets with an EF-proximity. ■

1.3. Strong inclusion.

Any proximity δ on X induces a binary relation over the powerset $\mathcal{P}(X)$, usually denoted as \ll_δ and named the *natural strong inclusion associated with δ* , by declaring that A is *strongly included* in B , $A \ll_\delta B$, when A is far from the complement of B , i.e., $A \not\phi X \setminus B$ [4]. In terms of strong inclusion associated with an EF-proximity δ , the *Efremovič property* for δ can be formulated as the betweenness property:

$$(EF2) \quad \text{If } A \ll_\delta B, \text{ then there exists some } C \text{ such that } A \ll_\delta C \ll_\delta B.$$

We conclude by emphasizing that a topological structure is based on the nearness between points and sets and a function between topological spaces is continuous provided it preserves nearness between points and sets, while a function between two proximity spaces is *proximally continuous*, provided it preserves nearness between sets. Of course, any proximally continuous function is continuous with respect to the underlying topologies.

2. DESCRIPTIVE INTERSECTION AND PETERS PROXIMITY

J.F. Peters made the first fusion of description with proximity, so passing from the classical spatial proximity to the recent more visual descriptive proximity which has a broad spectrum of applications [16, 20, 24, 23, 19, 22]

The starting idea is that two sets are near when the feature-values differences are so small so that they can be considered indistinguishable. He introduced the notion of descriptive intersection which, playing a similar role of set-intersection in the classical case, is crucial in our recent project to approach new forms of descriptive proximities. The mixture of description with proximity reveals an advantageous contamination.

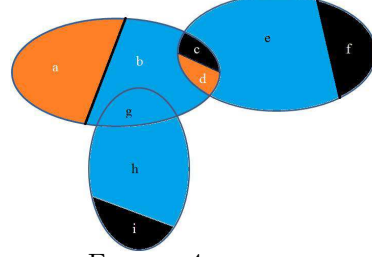


FIGURE 4.

$$\gamma_{\Phi} \neq \delta_{\Phi}$$

The descriptive intersection of two sets A, B is nonempty, provided there is at least one element in A with a description that matches the description of at least one element in B . The sets A, B cannot share any point in common but they can have a nonempty descriptive intersection.

Example 2.1. Let X be \mathbb{R}^2 and $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a probe that associate to each point its RGB-color. In Fig. 4, consider sets A, B, C and their subsets $a, b, c, d, e, f, g, h, i$. Observe that $\Phi(A) \cap \Phi(B)$ is given by colors black and red, so $\Phi^{-1}(\Phi(A) \cap \Phi(B)) = a \cup c \cup d \cup f \cup i$ and $A \underset{\Phi}{\cap} B = \Phi^{-1}(\Phi(A) \cap \Phi(B)) \cap (A \cup B) = a \cup c \cup d \cup f$. Then $A \underset{\Phi}{\cap} B \supseteq A \cap B = c \cup d$ and $A \underset{\Phi}{\cap} B \subseteq A \cup B$. ■

The first natural descriptive proximity, which we decided to call Peters proximity and to denote as π_{Φ} , declares two sets *descriptively near* iff their descriptions intersect. Or in other words:

Let X be a non-empty set, A and B be subsets of X , and $\Phi : X \rightarrow \mathbb{R}^n$ be a probe, then: $A \pi_{\Phi} B \Leftrightarrow \Phi(A) \cap \Phi(B) \neq \emptyset$. namely, Peters proximity π_{Φ} , which is the Φ -pull back of the discrete proximity.

Theorem 2.2. Peters proximity is an Efremovič proximity, whose underlying topology is R_0 and Alexandroff. Furthermore, π_{Φ} is T_0 , then T_2 , iff the probe Φ is injective.

Recall that a topological space has the *Alexandroff property* iff any intersection of open sets is in turn open [1].

It is easily seen that we can rewrite the previous definition by using Φ -saturation of sets.

Remark 2.3. Recall that a set A is called Φ -saturated if and only if $\Phi^{-1}(\Phi(A)) = A$. ■

Proposition 2.4. Let X be a non-empty set, A be subset of X , and $\Phi : X \rightarrow \mathbb{R}^n$ be a probe. Then A is closed in the topology induced by π_{Φ} , $\tau(\pi_{\Phi})$, if and only if it is Φ -saturated. Moreover $\tau(\pi_{\Phi})$ is disconnected.

Proof. The proof of the first part comes from the following equivalences $x \in Cl_{\pi_{\Phi}}(A) \Leftrightarrow x \pi_{\Phi} A \Leftrightarrow \Phi(x) \in \Phi(A) \Leftrightarrow x \in \Phi^{-1}(\Phi(A))$. To see that $\tau(\pi_{\Phi})$ is disconnected

consider $\Phi^{-1}(\Phi(x))$. This is a closed set being equal to $Cl\pi_\Phi(x)$, but at the same time it is open because its complement is given by $\bigcup\{\Phi^{-1}(\Phi(y)) : \Phi(y) \neq \Phi(x)\}$ and it is closed in its turn being Φ -saturated. \square

If we consider the relation on X given by $x\mathcal{R}_\Phi y \Leftrightarrow \Phi(x) = \Phi(y)$, then we have an equivalence relation whose classes are of type $[x] = \Phi^{-1}(\Phi(x))$, where $x \in X$. So two subsets of X , A and B , are π_Φ -near if and only if they intersect a same class of the partition induced by \mathcal{R}_Φ .

3. DESCRIPTIVE PROXIMITIES

Peters proximity is a link between nearness or overlapping of descriptions in the codomain \mathbb{R}^n with relations on pairs of subsets on the domain of codification. But Peters proximity π_Φ might be considered in some cases too strong or in some other ones too weak. So, by relaxing or stressing π_Φ , we obtain general forms of descriptive proximities, that can work better than it in particular settings. Since, from a spatial point of view, classical proximity is a generalization of the set-intersection, in our treatment we choose Peters proximity as the unique separation element between two different broad classes of descriptive proximities. If we entrust the descriptive intersection with the same role of the set-intersection in the classical case we get the following two options: descriptive intersection versus descriptive proximity, *i.e.*,

First option: weaker form:

$$A \underset{\Phi}{\cap} B \neq \emptyset \Rightarrow A \delta_\Phi B$$

Second option: stronger form:

$$A \delta_\Phi B \Rightarrow A \underset{\Phi}{\cap} B \neq \emptyset.$$

3.1. Weaker form.

This is the case in which two sets having nonempty descriptive intersection are descriptively near: $A \underset{\Phi}{\cap} B \neq \emptyset \Rightarrow A \delta_\Phi B$.

Let X be a non-empty set, A, B, C be subsets of X , and $\Phi : X \rightarrow \mathbb{R}^n$ be a probe. The relation δ_Φ on $\mathcal{P}(X)$, the powerset of X , is a *Čech Φ -descriptive proximity* iff the following properties hold:

- D_0): $A \delta_\Phi B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$
- D_1): $A \delta_\Phi B \Leftrightarrow B \delta_\Phi A$
- D_2): $A \underset{\Phi}{\cap} B \neq \emptyset \Rightarrow A \delta_\Phi B$
- D_3): $A \delta_\Phi (B \cup C) \Leftrightarrow A \delta_\Phi B$ or $A \delta_\Phi C$

If, additionally:

$$D_4): A \delta_\Phi B \text{ and } \{b\} \delta_\Phi C \text{ for each } b \in B \Rightarrow A \delta_\Phi C$$

holds, then δ_Φ is a *Lodato Φ -descriptive proximity* [16, §4.15.2,p.155].

Furthermore, if the following property holds:

$A \not\delta_\Phi B \Rightarrow \exists E \subset X$ such that $A \not\delta_\Phi E$ and $X \setminus E \not\delta_\Phi B$, then δ_Φ is an *EF Φ -descriptive proximity*.

We explicitly observe that descriptive axioms D_0 through D_4 are formally the same as in the classical definition with the set-intersection replaced by the Φ -intersection.

3.2. The underlying topology.

As in the classical case, for any descriptive proximity δ_Φ and for each subset A in X we define the Φ -descriptive closure of A as :

$$Cl_\Phi(A) =: \{x \in X : x \delta_\Phi A\}$$

Theorem 3.1. *The closure operator Cl_Φ is a Kuratowski operator iff δ_Φ is a Lodato Φ -descriptive proximity.*

Proof. Let $A, B, C \subset Cl_\Phi(D)$ and let δ_Φ is a Lodato Φ -descriptive proximity. The descriptive forms of P_0 - P_3 of Lodato proximity for δ_Φ are satisfied for A, B, C , if and only if Cl_Φ is a Kuratowski operator. \square

3.3. Examples.

Peters proximity is the Φ -pull back of the set-intersection. The set-intersection can be considered in two different aspects. It is the finest proximity on one side and the weakest overlap relation on the other side. So, to construct significant examples of descriptive proximities weaker than the Peters proximity we have two possible approaches: the proximal approach, which arises when looking at the the set-intersection as a proximity; the overlap approach, when looking at the set-intersection as an overlap relation.

3.4. Proximity approach. Let X be a nonempty set, A and B be subsets of X , $\Phi : X \rightarrow \mathbb{R}^n$ be a probe and δ be a proximity on \mathbb{R}^n . Then, if we define δ_Φ as follows:

$$A \delta_\Phi B \Leftrightarrow \Phi(A) \delta \Phi(B)$$

we get a descriptive proximity. The descriptive proximity δ_Φ and the standard proximity δ are very close to each other absorbing and transferring their own similar properties to the other.

Theorem 3.2. *The proximity δ is a Čech, Lodato or an EF-proximity iff, for each description Φ , δ_Φ is a Čech, Lodato or an EF Φ -descriptive proximity.*

Proof. We consider only classical EF proximity vs. EF Φ -descriptive proximity. The equivalence between the two EF holds when the previous axioms hold. \square

Observe that, given a proximity δ on \mathbb{R}^n , δ_Φ is the coarsest proximity on X for which the probe Φ is proximally continuous, i.e. $A \delta_\Phi B \Rightarrow \Phi(A) \delta \Phi(B)$ [12, §1.7, p. 16].

Of course, the prototype is the Peters proximity when \mathbb{R}^n is equipped with the discrete proximity. In this case the $Cl_\Phi(A)$ is the Φ -preimage of $\Phi(A)$.

Another significant example is the fine Lodato descriptive proximity.

When \mathbb{R}^n is equipped with the Euclidean topology, the finest Lodato proximity δ_0 is an EF-proximity. The relative descriptive proximity δ_Φ^0 , the fine Lodato descriptive proximity, is in its turn an EF-descriptive proximity.

The fine Lodato descriptive proximity:

$$A \delta_\Phi^0 B \Leftrightarrow Cl_E(\Phi(A)) \cap Cl_E(\Phi(B)) \neq \emptyset$$

Conjecture 3.3. *The fine Lodato descriptive proximity is the finest one among all "general" Lodato descriptive proximities as in the classical case.* ■

Based on the definition in [19], we can also consider the descriptive closure of a set

$$Cl_{\Phi}A = \{x : x\delta_{\Phi}A\} = \{x : \Phi(x) \in Cl_E(\Phi(A))\}$$

We prove now that we can re-write the fine descriptive proximity in terms of descriptive closures.

Proposition 3.4. *Let X be a non-empty set, A and B be subsets of X , and $\Phi : X \rightarrow \mathbb{R}^n$ be a probe.*

$$A \delta_{\Phi}^0 B \Leftrightarrow Cl(\Phi(A)) \cap Cl(\Phi(B)) \neq \emptyset \Leftrightarrow Cl_{\Phi}A \cap Cl_{\Phi}B \neq \emptyset.$$

Proof. $Cl(\Phi(A)) \cap Cl(\Phi(B)) \neq \emptyset \Leftrightarrow \exists y \in \Phi(X) : y \in Cl(\Phi(A)) \cap Cl(\Phi(B)) \Leftrightarrow y \delta \Phi(A) \text{ and } y \delta \Phi(B) \Leftrightarrow \exists x \in X : y = \Phi(x), \Phi(x) \delta \Phi(A) \text{ and } \Phi(x) \delta \Phi(B) \Leftrightarrow \exists x \in Cl_{\Phi}A \cap Cl_{\Phi}B \Leftrightarrow Cl_{\Phi}A \cap Cl_{\Phi}B \neq \emptyset.$ □

When requiring $A \pi_{\Phi} B$, we look at the match of the entire feature vectors on points of A and B . But, it can be useful to consider a fixed part of the vector of feature values. In this way descriptive nearness of sets can be established on a partial match of descriptions. To achieve this result, we introduce:

Definition 3.5. (β_{Φ}) .

Let X be a non-empty set, A and B be subsets of X , and $\Phi : X \rightarrow \mathbb{R}^n$ be a probe. We define

$$A \beta_{\Phi} B \Leftrightarrow \Phi_i(A) \cap \Phi_i(B) \neq \emptyset, \forall i = 1, \dots, n$$

Further, by generalizing β_{Φ} by composing the probe Φ with the projection π_m :

$$x \longrightarrow (\pi_m \circ \Phi)(x) = (\phi_1(x), \dots, \phi_m(x)).$$

we have:

Definition 3.6. (η_{Φ}) .

$$A \eta_{\Phi} B \Leftrightarrow \pi_m(\Phi(A)) \cap \pi_m(\Phi(B)) \neq \emptyset.$$

Proposition 3.7. *The relation η_{Φ} , then the relation β_{Φ} , is a Φ -descriptive EF-proximity.* ■

Example 3.8. *For an illustration of Prop. 3.7, see Fig. 5.*

Remark 3.9. *The topology associated with β_{Φ} is defined by the Kuratowski operator $Cl_{\beta_{\Phi}}$:*

$$x \in Cl_{\beta_{\Phi}}(A) \Leftrightarrow x \in \bigcap_{i=1, \dots, n} \Phi_i^{-1}(\Phi_i(A)) \quad \blacksquare$$

A third kind of descriptive relation, but not a descriptive nearness, defined by probes and intersection is given as follows.

Definition 3.10. (γ_{Φ}) .

Let X be a non-empty set, A and B be subsets of X , and $\Phi : X \rightarrow \mathbb{R}^n$ be a probe. We define $A \gamma_{\Phi} B \Leftrightarrow \exists i \in \{1, \dots, n\} : \Phi_i(A) \cap \Phi_i(B) \neq \emptyset.$

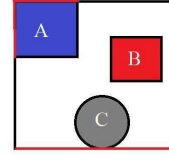
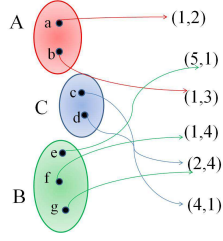


FIGURE
5.
 $\pi_{\Phi} \Rightarrow \beta_{\Phi}$

FIGURE 6. $\gamma_\Phi \neq \delta_\Phi$

The relation γ_Φ is not a descriptive proximity. We illustrate this by the following example based on Fig. 6.

Example 3.11. Let $A = \{a, b\}$, $C = \{c, d\}$, $B = \{e, f, g\}$. In this figure we have $\Phi_1(A) = \{1\}$, $\Phi_2(A) = \{2, 3\}$, $\Phi_1(C) = \{2, 4\}$, $\Phi_2(C) = \{4, 1\}$, $\Phi_1(B) = \{1, 2, 5\}$. So $A \gamma_\Phi B$ because $\Phi_1(A) \cap \Phi_1(B) \neq \emptyset$, and for each $x \in C$ $x \gamma_\Phi C$. But $A \not\gamma_\Phi C$ because $\Phi_1(A) \cap \Phi_1(C) = \emptyset$ and $\Phi_2(A) \cap \Phi_2(C) = \emptyset$. In other words γ_Φ is not a Lodato proximity [12, §3.1, p. 72]. ■

3.5. Overlapping approach.

Suppose that for any subset A of X a specific enlargement, $e(\Phi(A))$, of $\Phi(A)$ in \mathbb{R}^n can be associated with A and moreover, for any pair $A, B : e(\Phi(A)) \cup e(\Phi(B)) = e(\Phi(A) \cup \Phi(B))$ (*additivity*) and also $A \subseteq B \Rightarrow e(\Phi(A)) \subseteq e(\Phi(B))$, (*extensionality*) [5]. Then, if we put:

$$A \delta_\Phi B \text{ iff } e(\Phi(A)) \cap e(\Phi(B)) \neq \emptyset$$

we have:

Proposition 3.12. *The relation δ_Φ is a Φ -descriptive Lodato proximity.*

Proof. This result follows from the initial conditions. □

When choosing as $\epsilon > 0$ as level of approximation and as enlargement for any subset of \mathbb{R}^n the ϵ -enlargement, we have a peculiar case in the overlapping approach. It is not possible to remove additivity or extensionality as the following geometric example, related to the affine structure of \mathbb{R}^n , proves:

$$A \delta_\Phi B \Leftrightarrow \text{conv}(\Phi(A)) \cap \text{conv}(\Phi(B)) \neq \emptyset,$$

where $\text{conv}(\Phi(A)) = \text{minimal convex set containing } \Phi(A)$. The above relation verifies the properties D_0, D_1, D_2, D_4 but only one way in D_3 .

3.6. Second option: stronger form. This is the case in which two sets descriptively near have a nonempty descriptive intersection: $A \delta_\Phi B \Rightarrow A \cap_\Phi B \neq \emptyset$.

Let X be a non-empty set, A, B, C be subsets of X and $\Phi : X \rightarrow \mathbb{R}^n$ be a probe.

The relation δ_Φ^\wedge on $\mathcal{P}(X)$ is a Φ -descriptive Lodato strong proximity [21] iff the following properties hold:

- (S₀): $A \delta_\Phi^\wedge B \Rightarrow A \neq \emptyset \text{ and } B \neq \emptyset$
- (S₁): $A \delta_\Phi^\wedge B \Leftrightarrow B \delta_\Phi^\wedge A$
- (S₂): $A \delta_\Phi^\wedge B \Rightarrow A \cap_\Phi B \neq \emptyset$

$$(S_3): A \overset{\mathfrak{M}}{\delta}_{\Phi} (B \cup C) \Leftrightarrow A \overset{\mathfrak{M}}{\delta}_{\Phi} B \text{ or } A \overset{\mathfrak{M}}{\delta}_{\Phi} C$$

$$(S_4): A \overset{\mathfrak{M}}{\delta}_{\Phi} B \text{ and } \{b\} \overset{\mathfrak{M}}{\delta}_{\Phi} C \text{ for each } b \in B \Rightarrow A \overset{\mathfrak{M}}{\delta}_{\Phi} C.$$

As an example, when we can distinguish a significant subset $S \subseteq \Phi(X)$, we can put: $A \overset{\mathfrak{M}}{\delta}_{\Phi} B$ if and only if $\Phi(A)$ shares some common point with $\Phi(B)$ belonging to S , and obtain a strong Φ -descriptive proximity.

REFERENCES

1. P. Alexandroff and H. Hopf, *Topologie*, Springer-Verlag, Berlin, 1935, xiii+636pp.
2. V.A. Efremovič, *Infinitesimal spaces I (Russian)*, Doklady Akad. Nauk SSSR. N.S. 31 **76** (1951), 341–343, MR0040748.
3. ———, *The geometry of proximity(Russian)*, Mat. Sbornik. N.S. 31 **73** (1952), 189–200, MR0055659.
4. A. Di Concilio, *Proximity: A powerful tool in extension theory, functions spaces, hyperspaces, boolean algebras and point-free geometry*, Beyond Topology, AMS Contemporary Mathematics 486 (F. Mynard and E. Pearl, eds.), Amer. Math. Soc., 2009, MR2521943, pp. 89–114.
5. ———, *Point-free geometries: Proximities and quasi-metrics*, Math. in Comp. Sci. **7** (2013), no. 1, 31–42, MR3043916.
6. C. Guadagni, *Bornological convergences on local proximity spaces and ω_μ -metric spaces*, Ph.D. thesis, Università degli Studi di Salerno, Salerno, Italy, 2015, Supervisor: A. Di Concilio, 79pp.
7. C. Kuratowski, *Topologie i*, Państwowe Wydawnictwo Naukowe, Warsaw, 1958, XIII + 494pp.
8. K. Kuratowski, *Introduction to calculus*, Pergamon Press, Oxford, UK, 1961, 316pp.
9. M.W. Lodato, *On topologically induced generalized proximity relations*, ph.d. thesis, Rutgers University, 1962, supervisor: S. Leader.
10. ———, *On topologically induced generalized proximity relations i*, Proc. Amer. Math. Soc. **15** (1964), 417–422, MR0161305.
11. ———, *On topologically induced generalized proximity relations ii*, Pacific J. Math. **17** (1966), 131–135, MR0192470.
12. S.A. Naimpally and J.F. Peters, *Topology with applications. topological spaces via near and far*, World Scientific, Singapore, 2013, xv + 277 pp, Amer. Math. Soc. MR3075111.
13. S.A. Naimpally and B.D. Warrack, *Proximity spaces*, Cambridge Tract in Mathematics No. 59, Cambridge University Press, Cambridge, UK, 1970, x+128 pp., Paperback (2008), MR2573941.
14. M. Pavel, *Fundamentals of pattern recognition*, 2nd ed., Marcel Dekker, Inc., N.Y., U.S.A., 1993, xii+254 pp. ISBN: 0-8247-8883-4, MR1206233.
15. J.F. Peters, *Near sets. General theory about nearness of sets*, Applied Math. Sci. **1** (2007), no. 53, 2609–2629, MR2380165.
16. ———, *Topology of digital images - visual pattern discovery in proximity spaces*, Intelligent Systems Reference Library, vol. 63, Springer, 2014, xv + 411pp, Zentralblatt MATH Zbl 1295 68010.
17. ———, *Proximal Voronoï regions, convex polygons, & Leader uniform topology*, Advances in Math.: Sci. J. **4** (2015), no. 1, 1–5.
18. ———, *Visibility in proximal Delaunay meshes and strongly near Wallman proximity*, Advances in Math.: Sci. J. **4** (2015), no. 1, 41–47.
19. ———, *Computational proximity. Excursions in the topology of digital images.*, Springer, Berlin, 2016, Intelligent Systems Reference Library, 102.
20. J.F. Peters and C. Guadagni, *Strong proximities on smooth manifolds and Voronoï diagrams*, Advances in Math.: Sci. J. **4** (2015), no. 2, 91–107.
21. ———, *Strongly near proximity and hyperspace topology*, arXiv **1502** (2015), no. 05913, 1–6.
22. ———, *Strongly proximal continuity & strong connectedness*, Topology and Its Applications **204** (2016), 41–50.
23. J.F. Peters and R. Hettiarachchi, *Proximal manifold learning via descriptive neighbourhood selection*, Applied Math. Sci. **8** (2014), no. 71, 3513–3517.
24. J.F. Peters and S.A. Naimpally, *Applications of near sets*, Notices of the Amer. Math. Soc. **59** (2012), no. 4, 536–542, DOI: dx.doi.org/10.1090/noti817, MR2951956.

- 25. E. Čech, *Topological spaces*, John Wiley & Sons Ltd., London, 1966, fr seminar, Brno, 1936-1939; rev. ed. Z. Frolik, M. Katětov.
- 26. S. Willard, *General topology*, Dover Pub., Inc., Mineola, NY, 1970, xii + 369pp, ISBN: 0-486-43479-6 54-02, MR0264581.

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